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MODULAR INFLATED SHELLS – A COMPUTATIONAL APPROACH

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Summary. This paper presents the results of the first phase of work on developing computational model of inflated modular shells. This type of bending active structures is composed of relatively small, modular inflatable cushions combined with cables and cross-braces. The structure is self-erecting. Introduction of tension to the cable gives it shape and load carrying capacity. The initial experiments confirmed their technological feasibility. At the present stage the relationship between tensile force in the cable, and the initial deformation of the structure were analyzed. Modular shell was approximated by an elastic rod. The impact of the internal structure of the coating on the computational model is the subject of further research.

1 INTRODUCTION

The proposal to use relatively small, modular inflatable cushions for construction of covers of different span has been presented previously[1,3]. The modular inflated shells are composed of relatively small inflated cushions combined with cables and cross-braces.

This solution allows construction of single or double curved shells, which stiffness can be adjusted by changing their structural height e.g. by changing the thickness of the cushions or length of cross-braces, and also by changing the pre-tension force in the cables. Thickness variation along the entire span of the structure enables not only changing of the stiffness, but also adjustment of the initial curvature, which may also be variable. This allows forming large and complex structures tailored to meet specific and even rapidly changing needs.

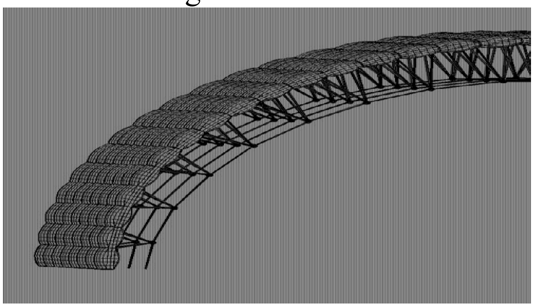
The object can be open at the sides and the openings in its surface are allowed. The structure is not sensitive on local damage of elements – even if many cushions are out of service whole the structure can be safely used. Due to internal fit out, the structure can be easily maintained. This type of structures can be used for many military and civil applications, where a fast assembled and adaptable solution is required.

The initial experiments confirmed the technological feasibility of this type of structures[2]. However, some technological problems have been revealed that needed to be analyzed with use of the large scale physical models[4].

Currently, an attempt was made to create a calculation model to describe them[5]. This paper presents the results of the first phase of work, covering structure of a single curvature. It was assumed a beam-like structure between a roller support and a simple support. The end at the roller support is attached to a cable, which is pulled horizontally through a hole in the simple support. After the structure is deformed to its final position, the pulling cable is clamped at the simple support end. Then the equations describing the problem are derived, based on the theory of buckling with large deformations. At this stage the problem of modelling cables sliding through the nodes connecting them with cross braces has not been yet analyzed. This will be the subject of the next phase of work.

2. FORMULATION OF THE PROBLEM

Considered structure consists of the three groups of elements: modular inflated cushions, tension cables and cross-braces, Figure 1, left. The latter are optional and are used to increase the structural height. This can also be done by increasing the thickness of the cushions, in the whole structure or part thereof (variable rigidity of the structure). The structure may be shaped as an arc or a single or double curved shell.



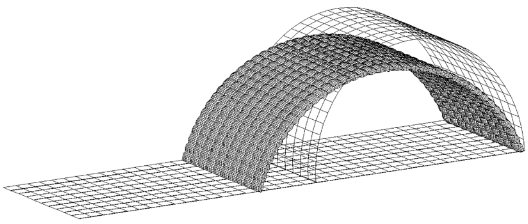


Figure 1: Typical configuration of the modular inflated shell (left); process of self-erection (right)

The flat structure is assembled at ground level as a near mechanism. It is stabilized and finally shaped in the self-erection process. The essence of the process is the introduction into the structure forces that cause its large deformation (uplift) and give the rigidity. The forces are introduced by pulling the bottom tension cable, thus reducing the distance between the supports. The system becomes bending-active. Figure 1, right, shows a general idea of this process.

The formulation of the calculation model of bending-active inflated shell is a complex problem. To solve it, a simpler problem has been conceived, in which the beam-like structure is substituted for modular inflated shell. This allows figuring out the problem from an analytical view point, without being concerned about the details of the construction of the real structure.

We consider a beam-like structure between a roller support *A* and a simple support *B*. The end at the roller support is attached to a cable, which is pulled horizontally through a hole drilled in the simple support, and, after the beam-like structure has deformed to a maximum height h, the pulling cable is clamped at the simple support end.

A free body diagram of the system can be drawn, cutting out the cable, and assuming weight is not a significant force. It shows applied horizontal tension load T at the location of the roller support pointing toward the simple support, and the equal and opposite reaction R_H load at the simple support. Vertical reactions at the two supports are neglected, as they have to be equal and opposite, but must be zero because there is no external agent capable of counteracting the resulting couple moment. Effectively, the free body diagram is the same as that of the Euler column, Figure 2.

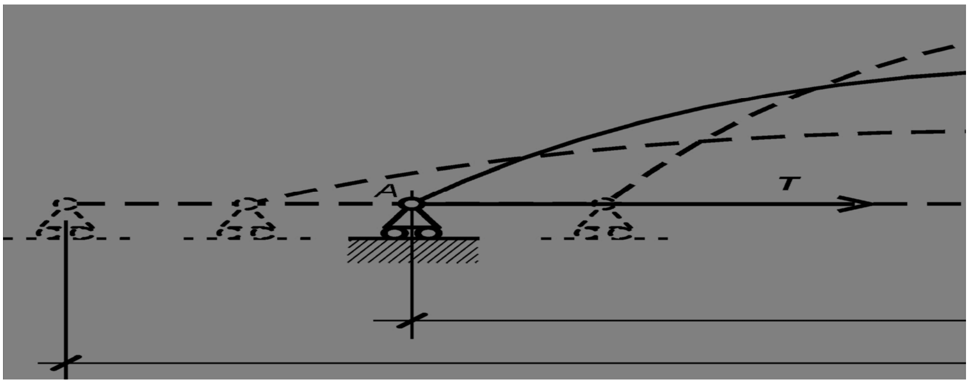


Figure 2: A free body diagram of the considered system

If we make a cut at some coordinate x along the length from the simple support B to the roller support B and take a free-body diagram of one half of the body, the moment equilibrium equation is:

$$M\left(x\right)+Ty=0 \tag{1}$$

Here, *M(x)* is the internal moment, T is the tension in the cable, and y is the amount of deflection at position x. From this point on, further analysis will be carried out in two steps: first, assuming that the deflections are "small enough", then for large deflections.

3. INITIAL APPROACH – SMALL DEFLECTION SOLUTION

Assuming a linearly elastic material, the equation (1) for the bending structure can be written as:

$$EI\kappa+Ty=0 \tag{2}$$

Here, according to Euler-Bernoulli law, κ is the curvature of the elastic curve of the structure:

$$\kappa=\frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}} \tag{3}$$

If θ is introduced as the first derivative of y: $\theta=dy/dx$, and is then assumed to be an explicit function of y, then:

$$\kappa=\frac{\theta\frac{d\theta}{dy}}{\left[1+\theta^2\right]^{3/2}} \tag{4}$$

When deflections are small, the denominator of the above expression approaches a value of 1 and κ is approximated by the second derivative of *y*.

$$\kappa = \frac{d\theta}{dx} = \frac{d\theta}{dy}\frac{dy}{dx} = \theta\frac{d\theta}{dy} \tag{5}$$

Substituting equation (5) into equation (2) and separating variables yields:

$$\theta d\theta = -\lambda_1^2 y dy \tag{6}$$

Here, λ_1 is defined as:

$$\lambda_1 = \sqrt{\frac{T}{EI}} \tag{7}$$

It should be noted that from this point forward that λ_1 does not have a strong relationship with *T* as implied by equation (7), but is used simply as a mechanism to derive a compatible shape. The value of *T* is determined through the following methodology.

From the equations (2) and (5) it follows that for "small enough" deflections, moment *M* is related to the second derivative of y times *EI*. Given the form of the differential equation, the boundary conditions and the desired result for maximum deflection, a guess for the form of *y* is:

$$y(x) = h\sin\frac{\pi x}{\ell} \tag{8}$$

Where, l is the current distance between the supports, and is considered an unknown in the problem. The beam-like structure has a net compression transmitted through it; however, at this stage we will ignore the deformation associated with this compression. Thus, for a given value of *h*, the variable *l* can be solved using the equation:

$$L = \int_0^{\ell}\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx \tag{9}$$

Here, *L* is the length of the beam-like structure when undeformed.

The strain energy of the beam-like structure is formulated by using equations (1) and (8), and on the base of the Clapeyron theorem can be shown to be:

$$U = \frac{T^2 h^2 \ell}{4EI} \tag{10}$$

Applying the Castigliano theorem, one now takes the derivative of the strain energy with respect to *T* to yield the movement of the roller support toward the simple support, which equals *L – l*. Solving for *T* yields:

$$T = \frac{2EI}{h^2}\left(\frac{L}{\ell} - 1\right) \tag{11}$$

Now we can return to equation (6) in order to analyze on this basis the elastic curve of the structure. Integrating equation (6) yields:

$$\frac{1}{2}\theta^2 = -\frac{1}{2}\lambda_1^2 y^2 + C_1 \tag{12}$$

Here, C_1 is an arbitrary constant. Assuming symmetry of deformation and maximum deflection h, then $\theta = 0$ when $y = h$. Using this boundary condition, solving for C_1 and plugging back into equation (12) yields:

$$\frac{1}{2}\theta^2 = \frac{1}{2}\lambda_1^2\left(h^2 - y^2\right) \tag{13}$$

Manipulating equation (13), remembering the definition of θ, and separating variables yields:

$$\frac{dy}{\lambda_1\sqrt{h^2 - y^2}} = dx \tag{14}$$

Here, we incorporate the boundary condition that when $x = 0$ then $y = 0$. This can be done by taking definite integrals from 0 to y of the left side of equation (14) and from 0 to x on the right side of equation (14).

It may be advantageous to non-dimensionalize at this point, defining η and ξ as:

$$\eta = \frac{y}{h}; \quad \xi = \frac{x}{\ell} \tag{15}$$

We then substitute into equation (14) and manipulate to obtain:

$$\frac{d\eta}{\sqrt{1-\eta^2}} = \lambda d\xi \tag{16}$$

Here, the non-dimensional parameter λ is λ_1 multiplied by l. Integrating both sides and rearranging yields:

$$\eta = \sin\lambda\xi \tag{17}$$

When $\xi = ½$, $\eta = 1$; thus, the simplest assignment for λ is π. Substituting for the non-dimensional variables as defined in equation (15) gives us the half-sine wave shape, which is then used to define the strain energy.

4. EXTENSION OF THE SOLUTION TO LARGE DEFLECTIONS

Dealing with large deflections is based on a similar approach. Here, the curvature is described by full expression given in equations (3) and (4).

The latter term is used on the left-hand side of equation (6) after separation of variables. Following the same procedure as above, and defining $\beta = h / l$ yields:

$$\frac{\left[1-\frac{1}{2}(\lambda\beta)^2\left(1-\eta^2\right)\right]d\eta}{\sqrt{1-\eta^2}\sqrt{1-\left(\frac{\lambda\beta}{2}\right)^2\left(1-\eta^2\right)}} = \frac{M[(\lambda\beta);\eta]}{\sqrt{1-\eta^2}}d\eta = \lambda d\xi \tag{18}$$

Where M is a multiplicative factor, which is a function of $\lambda\beta$ and η:

$$M[(\lambda\beta);\eta] = \frac{1-\frac{1}{2}(\lambda\beta)^2\left(1-\eta^2\right)}{\sqrt{1-\left(\frac{\lambda\beta}{2}\right)^2\left(1-\eta^2\right)}} \tag{19}$$

It can be seen that for "small" values of β that the left-hand side of equation (18) reduces to that of equation (16).

A few comments should be made about the multiplicative factor M. First, looking at the square root in the denominator, it is seen that $\lambda\beta$ can never exceed 2. Looking at the numerator, it is seen that if $\lambda\beta$ exceeds $\sqrt{2}$ that negative contributions will be made to the integral. Physically, this implies that the shape of the structure will "double back" in the x coordinate.

5 DISCUSSION OF THE RESULTS

In order to understand the relations expressed by equation (18), calculations were carried out to allow presentation of particular quantities versus $\lambda\beta$. The problem is thus laid out as follows:

- Choose a value of $\lambda\beta$
- Integrate the left hand side; the result is λ / 2; thus solving not only for λ, but β as well
- Assuming a value of λ is found, determine the shape by integrating equation (18) partially to generate a y vs. x curve.
- Numerically integrate to find the strain energy in a manner similar to that presented in paragraph 3.
- Repeat this procedure for a family of values of $\lambda\beta$.

Equation (18) was solved using Simpson's Rule. Half-steps of 0.001 were taken from $\eta = 0$ to 0.998; 0.0001 from 0.998 to 0.9998; 0.00001 from 0.9998 to 0.99998; and 0.000001 from 0.99998 to 0.999998. For the last 0.000002 where the integrand becomes very large, it can be shown that the integral is to a high degree of precision $\sqrt{2(0.000002)}$ or 0.002. Note that M is essentially 1 for $\eta = 1$.

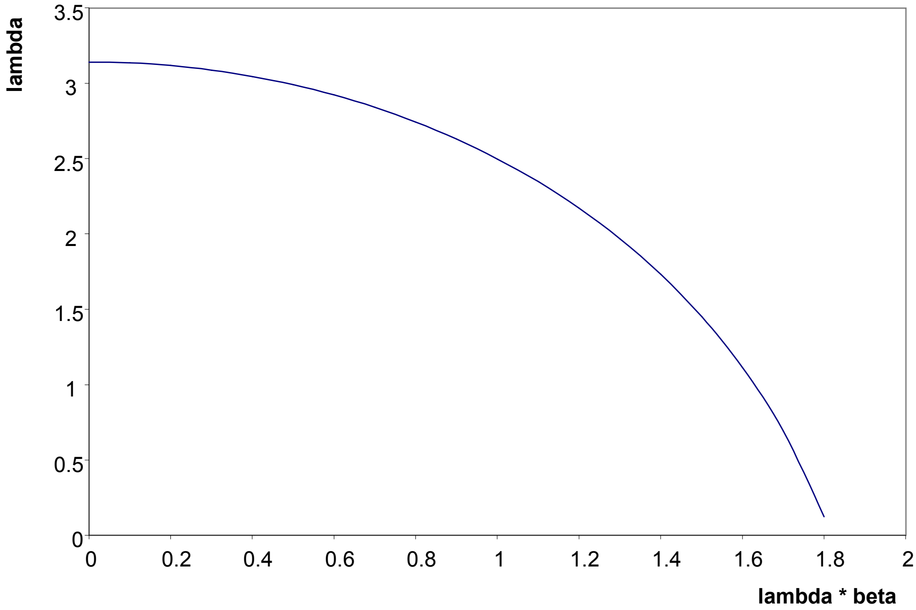


Figure 3: λ as a function of the $\lambda\beta$

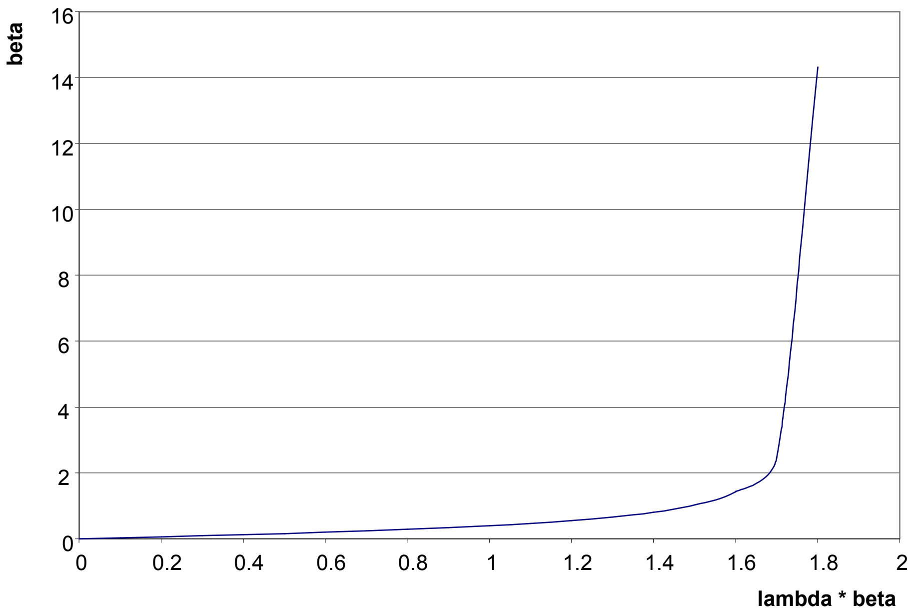


Figure 4: β as a function of the $\lambda\beta$

Figures 3 and 4 depict the results for λ and β. For Figure3, it should be noted that for $\lambda\beta = 0$ that $\lambda = \pi$ as expected. The value of λ then decreases as the non-linearity increases. It should be noted that somewhere between $\lambda\beta = 1.8$ and 1.9 that λ will go to 0; the physical meaning of this, and the effect on β are not known at this time.

One of the sub-computations is to determine l by setting the length of the deformed curve to the original length L. This relationship can be expressed as:

$$\int ds = \int \sqrt{(dx)^2 + (dy)^2} = L \tag{20}$$

Substituting the non-dimensional relationships of equation (15) and the definition of β yields:

$$\int d\sigma = \int \sqrt{(d\xi)^2 + (\beta d\eta)^2} = \frac{1}{\gamma} \tag{21}$$

Here, γ is defined as l / L and dσ is defined as ds / l. A plot of γ is presented in Figure 5.

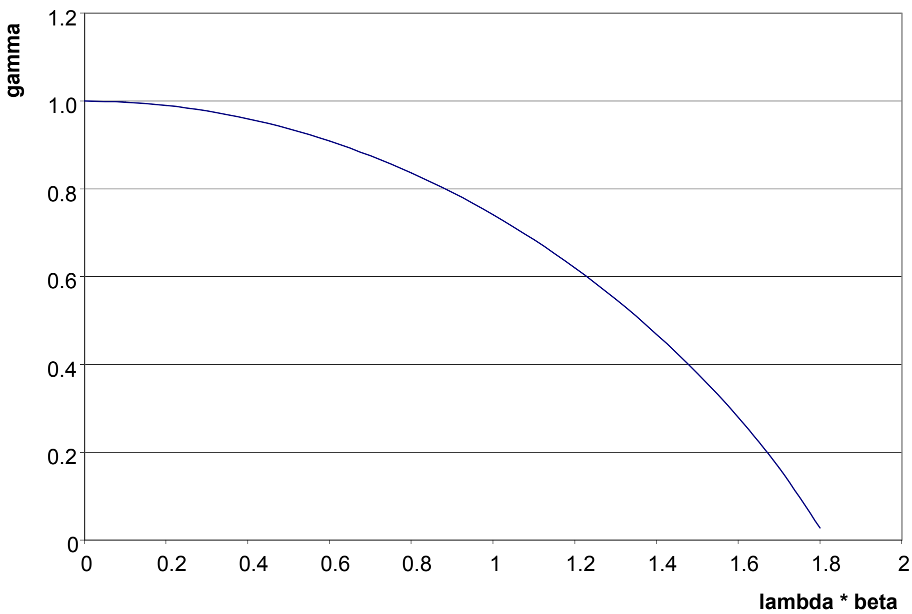


Figure 5: γ as a function of the $\lambda\beta$

It can be shown that for small values of $\lambda\beta$ that:

$$\gamma \approx 1 - \left(\frac{\lambda\beta}{2}\right)^2 \tag{22}$$

The integration of equation (18) is now used to generate η vs. ξ plots for selected values of $\lambda\beta$. In order to increase the physical meaning of these plots they are converted back to y and x, respectively, each non-dimensionalized by L, by multiplying η by $\gamma\beta$ and ξ by γ. These plots are displayed in Figure 6. It is important to note that the full shape of each curve includes a symmetrical segment about the right end of the curve as shown. It should also be noted that for $\lambda\beta$ approaching 0 the shape is a half sine wave (including the reflected portion of the curve) with infinitesimal height.

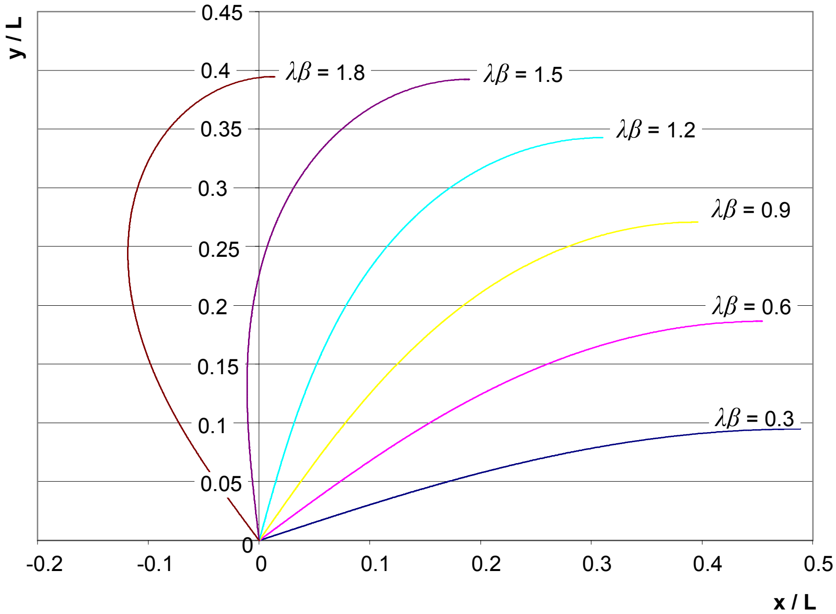


Figure 6: Shape curves for different values of $\lambda\beta$

The final set of calculations determines the tension T required to form the shapes noted. As discussed previously in paragraph 3, the tension can be determined by taking the derivative of the strain energy with respect to T. It can be shown that it is related to the shortening of the distance between the two supports by:

$$L-\ell=\frac{T}{EI}\int y^2 ds \tag{23}$$

Note that the integral is taken over the length of the structure, not just the x coordinate, to account for the large deformation. Non-dimensionalizing and re-arranging yields:

$$\tau=\frac{1}{2\mu(\gamma\beta)^2}\left[\frac{1}{\gamma}-1\right] \tag{24}$$

Here, τ represents non-dimensionalized tension defined as TL^2 / EI, and μ represents non-dimensional strain energy:

$$\mu=\int_0^{0.5}\eta^2 d\sigma \tag{25}$$

Figures 7 and 8 are plots of μ and τ, respectively, vs. $\lambda\beta$.

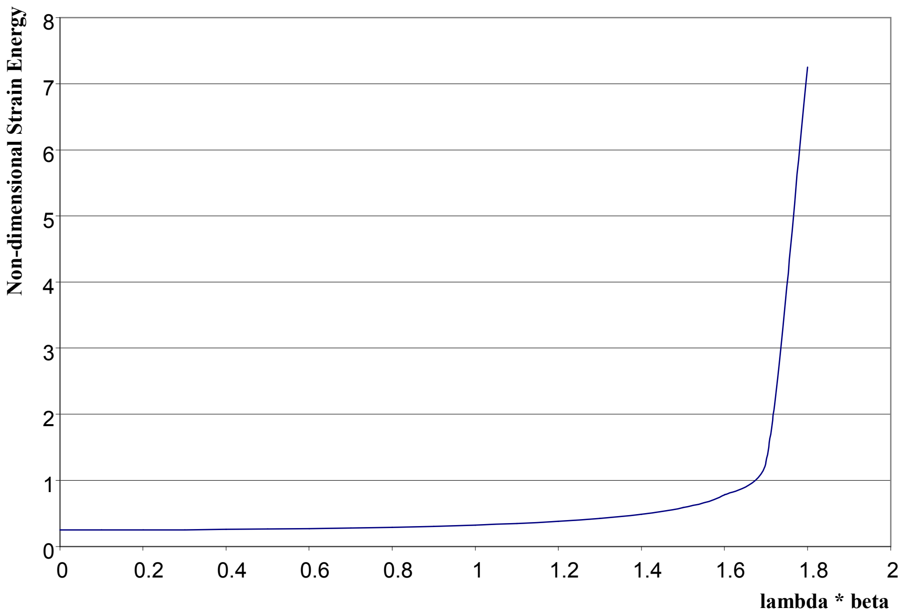


Figure 7: Non-dimensional strain energy μ as a function of the $\lambda\beta$

Of interest is the limit of τ as the non-linearity gets small. It should be noted that:

$$\mu = \int_0^{0.5} \sin^2 \pi\xi d\xi = 0.25 \tag{26}$$

Furthermore, if one substitutes the approximation of equation (22), the zero β terms cancel and:

$$\tau = \frac{\lambda^2}{8\mu\gamma^2} = \frac{\pi^2}{2} \tag{27}$$

This appears consistent with the rest of Figure 7. It should be noted that this is half of the classical Euler load for the structure.

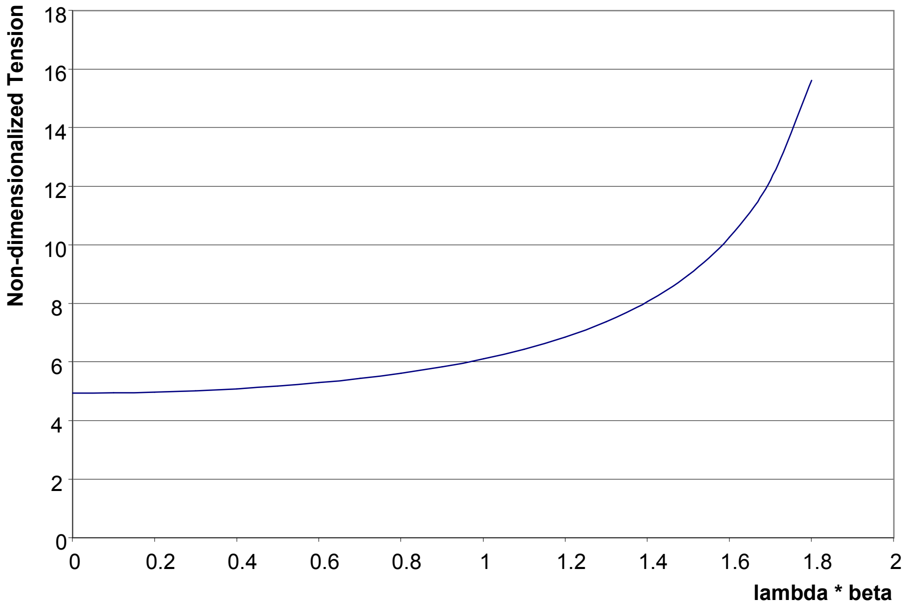


Figure 8: Non-dimesionalized tension τ as a function of the $\lambda\beta$

6. CONCLUSIONS

Presented in this paper attempt to computational modeling of modular inflated shell is based on a simplified physical model. The complex internal structure of the shell was approximated by an elastic rod. Issues of determining the flexural stiffness *EI* of the structure composed of inflated cushions were omitted. Similarly, were omitted issues of the interaction between the cushions-cross braces system and cable sliding through the nodes of the bottom chord. It should be noted that he analysis included only initial stage – self-erection of bending active structure. No cases of the external load were considered. These issues will be examined in further stages of work.

However, the analytical study carried out in paragraph 5 allowed making some general observations regarding the behavior of the structure. The conclusions of this analytical study are as follows:

- Unless initial imperfections in the straightness of the structure are taken into effect, no deformation of the structure will occur until the tension reaches a magnitude of half of the Euler buckling load. This is consistent qualitatively with experimental observation.
- As the tension increases, the two supports move closer to each other as expected. Eventually, the two supports will meet. No exploration of that event was pursued in this study.
- The parameter λ is associated with the shape of the deformed curve. It is somewhat related physically to the square root of τ times 2 times γ^2 though not in an exact way. Accordingly, this physical relationship will not be emphasized.
- The combined parameter $\lambda\beta$ has a mathematical limit of 2; however, there appears to be a physical limit imposed on the equations of somewhat greater than 1.8.

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